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# Inter-vehicle gap statistics on signal-controlled crossroads

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**Abstract**
We investigate a microscopical structure in a chain of cars waiting at a red signal on signal-controlled crossroads. A one-dimensional space-continuous thermodynamical model leading to an excellent agreement with the data measured is presented. Moreover, we demonstrate that an inter-vehicle spacing distribution disclosed in relevant traffic data agrees with the thermal-balance distribution of particles in the thermodynamical traffic gas (discussed in [1]) with a high inverse temperature (corresponding to a strong traffic congestion). Therefore, as we affirm, such a system of stationary cars can be understood as a specific state of the traffic sample operating inside a congested traffic stream.

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(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

The investigation of various transport systems is currently one of the prominent subjects of physics. The intention of such research is to describe these systems (or phenomena) quantitatively, create their appropriate models (theoretical or numerical), and finally obtain the exact or numerical outputs comparable to real situations. Higher aspirations of such research might be finding a certain connection among the different phenomena and revealing a possible universality.

Currently, one of the strongly accented fields is an investigation of queuing systems. Within this field many varied topics have been discussed, for example, a wide-ranging spectrum of vehicular traffic problems [2], pedestrian dynamics [3], escape panic [4], longitudinal parking of cars on a street [5, 6], parallel parking [7, 8] and public transport in some Latin America countries [9, 10]. All these subjects are closely connected to the random matrix theory, theory of chaos or theory of particle gases (see the references cited above). The main goal of this paper is to extend the set of queuing systems mentioned above by a stationary ensemble of cars waiting at a red signal on signal-controlled crossroads (see also [11]).

Moreover, we are aiming to create a one-dimensional model of point-like vehicles producing the same inter-vehicle gap distributions as those detected among cars standing on signal-controlled intersections. In the second part of this paper, we demonstrate that such a model can be interpreted (on a microscopical level) as a thermodynamical gas of dimensionless particles exposed to a thermal bath. This analogy allows us, as we assert, to find an exact form of relevant spacing distribution which can be consequently compared to the realistic gap statistics.

## 2. Describing the system

The traffic data analysed in this work were measured over a few days on a multi-lane intersection located near the center of Prague. This intersection is a constituent of an extensive network of roads and crossroads inside the internal metropolis and is therefore strongly saturated almost all day. Furthermore, the time interval between two green signals (on one crossroad) is very short, which means that some cars are not able to reach the threshold of the following intersection (during one green phase) and therefore have to wait for another green light. This fact finally leads to a substantial decrease in average velocity of vehicles moving between crossroads, i.e. one can observe the effects usually detected in a congested traffic regime (see [2]). Bumper-to-bumper distances $r_i$ between subsequent cars ($(i+1)$th and $i$th ones) waiting at a red signal (in one direction only) were measured. The data file contains 5022 digitally gauged events showing the mean inter-vehicle gap approximately equal to 149 cm. The clearances were measured directly using the laser technology.

More detailed statistical analysis uncovers that a probability density $p(r)$ for distance $r$ between neighbouring cars shows a similar behaviour to that investigated between the eigenvalues of random matrices (see [12]), zeros of Riemann zeta function (see [13]) or vehicles moving inside the traffic stream on the freeways (see [1]). Such a behaviour (see figure 2) demonstrates the presence of repulsive interactions among the elements in question. As is well known, a spacing distribution of non-interacting elements shows a different distribution, in concrete: *Poisson probability density*

$$p(r) = \exp[-r] \qquad (r \geqslant 0).$$

Since the traffic interaction (in the local sense, of course) is usually quantified as power-law repulsion among the successive vehicles (see [1] and [14]) let us suppose that a potential energy of the ensemble investigated reads as

$$U(r_1, r_2, \ldots, r_N) = \sum_{i=1}^{N} r_i^{-1}. \tag{1}$$

Herein we assume that the stationary traffic state analysed in this paper (i.e. the queue of waiting cars) is determined by the preceding process—traffic flow towards the intersection. Evidently, moving in the traffic sample the driver is interacting with other cars and optimizing his/her motion to reach the threshold of the crossroad as soon as possible and, at the same time, avoiding a crash with the preceding vehicle. Such behaviour corresponds to the thermodynamic effects governing the ensemble into a local thermal equilibrium (see [1] for details).

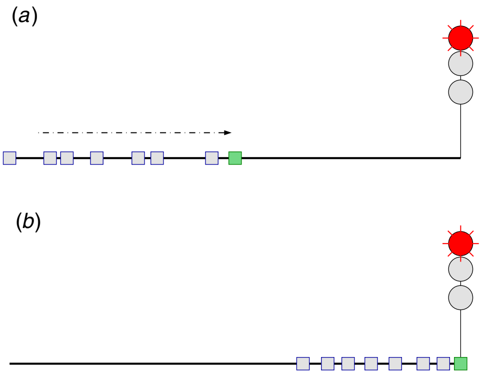


**Figure 1.** Graphic representation of the model. The upper subplot depicts the initial state of the numerical scheme described in the text whereas the bottom subplot demonstrates the final stationary state of the traffic sample, i.e. the state when the cars are waiting for green signal. We note that the squares represent the model particles with the leading car being picked out.

## 3. Modified Metropolis algorithm

Accepting the above-mentioned assumptions on thermodynamical aspects of the issue we formulate the following one-dimensional traffic model based on principles of statistical physics. Consider $N + 1$ point-like particles (cars) located randomly (or equidistantly if advantageous) on a line (or on a circle) so that the mean gap among them is one, i.e.

$$\sum_{i=1}^{N} r_i = N, \tag{2}$$

where $r_i$ represents the gap between $(i + 1)$th and $i$th particles. Thus, the ordered positions $x_1 > x_2 > \cdots > x_{N+1}$ constitute the initial state for our simulation (see figure 1). The particles move along the line (or along the circle) accepting the undermentioned rules until the leading car reaches a fixed point (the threshold of new crossroads). In accord with a realistic situation the overtaking cars are not permitted, i.e. the particles cannot change their order. Let $\beta_{\text{model}} \geqslant 0$ denote the inverse temperature specifying the measure of chaos inside the ensemble simulated. We assume $\beta_{\text{model}}$ to be the only significant parameter of the model. The car positions are repeatedly updated (we use 20 000 steps in our version) according to the following rules:

(1) The potential energy $U_0$ (using formula (1)) for the actual set of locations $\{x_1, x_2, \ldots, x_{N+1}\}$ is calculated.
(2) We pick an index $j \in \{1, 2, \ldots, N + 1\}$ at random.
(3) We draw a random number $\delta$ equally distributed in the interval $(0, 1)$.
(4) We compute an anticipated position $x'_j = x_j + \delta$ of the $j$th element. Because of singularity in the potential energy (1) the model particles cannot change their order. Therefore we accept $x'_j$ only if $x'_j < x_{j-1}$.
(5) We calculate a value of potential energy $U'$ determined for configuration $\{x_1, x_2, \ldots, x_{j-1}, x'_j, x_{j+1}, \ldots, x_{N+1}\}$.
(6) If $U' \leqslant U_0$ the $j$th particle position take on a new value $x'_j$. If $U' > U_0$ then the Boltzmann factor

$$w = \exp[-\beta_{\text{model}} \Delta U],$$

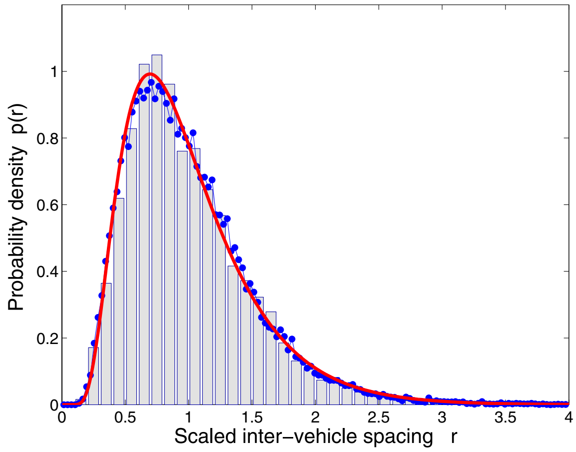


**Figure 2.** Inter-vehicle gap statistics $p(r)$. Bars represent the probability density for bumper-to-bumper distance among the cars waiting at a red signal on intersections (measured in Prague). Data were re-scaled so that the mean spacing is equal to one. Points display the optimized result of the numerical scheme (Metropolis algorithm) for $N = 100$ and $\beta_{\text{model}} = 1.45$. Finally, the curve displays distribution (3) for the fitted value $\beta_{\text{fit}} \approx 1.2488$ (obtained by the number variance test.)

where $\Delta U = U' - U_0$, should be compared with a random number $r$ equally distributed in $(0, 1)$. Provided that the inequality $w > r$ is fulfilled the $j$th particle position takes on the new value $x'_j$ too. Otherwise, the original configuration $\{x_1, x_2, \ldots, x_{N+1}\}$ remains unchanged.

The sketched procedure represents a modified Metropolis algorithm originally developed for chemistry purposes (in [15]). This algorithm belongs to the category of Monte Carlo simulations (see [17]) which have recently been used for numerical modelling of statistical systems (as demonstrated in [16], for example). The elaborated scheme of Metropolis ensures a relaxation of ensemble into a thermal-balance state when the energy fluctuates around a constant value being independent of initial configuration of particles (see figure 3). After reaching the thermal equilibrium (i.e. after approximately 5000 updates of configuration (Monte Carlo steps), as visible in figure 3) the ensemble lingers in this state until the simulation is interrupted. Then, as observed, corresponding probability density for inter-particle gaps depends on the inverse temperature $\beta_{\text{model}}$ only.

Our aim is to find the optimal value of inverse temperature $\beta_{\text{model}}$ so that the gap distribution $p(r)$ corresponds to that measured among the cars on crossroads. Using a $\chi^2$-method (i.e. minimizing the sum of squares-deviations between two distributions in question) one can find that optimal value of $\beta_{\text{model}}$ is approximately 1.45. Concretely, for a fixed value of $\beta_{\text{model}}$ the distribution $p(r)$ is obtained. Then the $\chi^2$-test between empirical data and $p(r)$ could be evaluated. The optimal value of $\beta_{\text{model}}$ is the one for which the corresponding sum of squares-deviations is minimal. To conclude, for value of $\beta_{\text{model}} = 1.45$ both processes (traffic and Metropolis procedures) generate practically the same gap distributions (see figure 2). Thus, the introduced procedure could represent a realistic model for behaviour of the cars in the vicinity of the chosen intersection.

## 4. Terminal state of thermodynamical traffic gas

As explored in papers [1, 14, 18], the traffic flow can be understood (on a microscopical level) as a thermodynamical gas of interacting cars exposed to a heat bath of inverse temperature $\beta$.

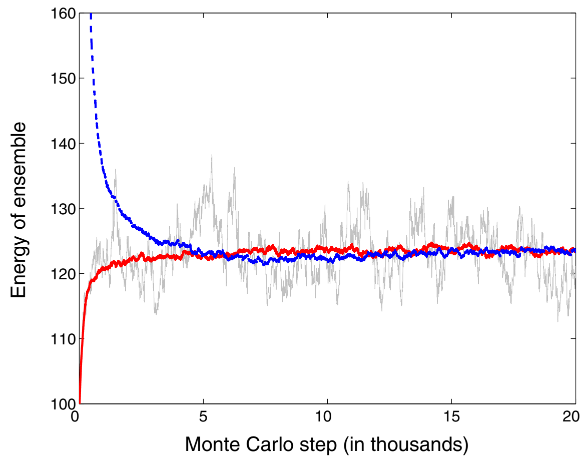


**Figure 3.** Relaxation of the system into the thermal equilibrium. Dashed and continuous lines (see the upper-left or lower-left corners, respectively) display the energy value (1) for $N = 100$ and $\beta_{\text{model}} = 1.45$ during the run of Metropolis procedure (having 20 000 steps) for random (or equidistant) initial locations of elements, respectively. Plotted is the average value (calculated for 100 repeated realizations of Metropolis). The grey curve represents the energy value (1) for one realization of Metropolis (when initial particle positions were chosen equidistantly).

Besides, the latter has an immediate relation to the traffic density. If accepting such an approach we describe the traffic ensemble (on the move) as a circular gas of point-like particles whose hamiltonian reads as

$$\mathcal{H} = \sum_i (v_i - \overline{v})^2 + \sum_i r_i^{-1},$$

where $v_i$ and $r_i$ represent an $i$th car velocity and gap to the previous car, respectively. Quantity $\overline{v}$ denotes the desired velocity of the ensemble. Then (see the exact calculation in [1]) the derived probability density $p_\beta(r)$ for a gap $r$ among the successive vehicles is

$$p_\beta(r) = A \exp[\beta r^{-1} - Br], \tag{3}$$

where the constants $A$ and $B$ are calculated via two normalization equations

$$\int_0^\infty p_\beta(r)\,\mathrm{d}r = \int_0^\infty r p_\beta(r)\,\mathrm{d}r = 1.$$

According to [1] the following relations hold true:

$$B \approx \beta + \frac{3 - \mathrm{e}^{-\sqrt{\beta}}}{2},$$

$$A \approx \frac{\sqrt{2\beta + 3 - \mathrm{e}^{-\sqrt{\beta}}}}{\sqrt{8\beta}\mathcal{K}_1(\sqrt{4\beta^2 + 6\beta - 2\beta\,\mathrm{e}^{-\sqrt{\beta}}})}.$$

Herein $\mathcal{K}_1$ stands for a Mac-Donald's function (modified Bessel's function of the second kind) of the first order.

Since the situation investigated in this paper is without any doubt the result of a preceding traffic flow (see [11]) it is meaningful to expect that the clearance distribution among the cars waiting at the red-light-signal will be of the form (3). Indeed, as confirmed by an appropriate statistical analysis of the collected data (discussed later) the measured gap statistics (see figure 2) corresponds to the probability density (3) if the inverse temperature $\beta$ of the thermodynamical model is

$$\beta_{\rm fit} \approx 1.2488. \tag{4}$$

We denote that this value has been determined by a more sophisticated method presented in the following section. In addition, a positive comparison between the corresponding gap distributions supports the hypothesis that traffic stream can be *locally* understood as a stochastic gas whose elements are repulsed by the forwardly-directed nearest-neighbour power-law potential depending on a reciprocal distance between successive gas elements. This correspondence, however, does not mean that traffic is a thermodynamical system, of course.

## 5. Testing the statistical variance of data

If trying to find a more robust argument for an assertion on statistical similarities between the process investigated and the traffic model we can apply some of the techniques originally developed for purposes of the random matrix theory (see the book [12]). A usual way to quantify the behaviour of variances among the statistical data is in applying so-called *number-variance test*. Such a test is defined as follows.

Consider $N$ spacings $r_1, r_2, \ldots, r_N$ between the successive vehicles (or particles of model) and suppose that the mean distance taken over the complete ensemble is re-scaled to one, i.e.

$$\sum_{i=1}^{N} r_i = N.$$

Dividing the interval $[0, N]$ into subintervals $[(k-1)L, kL]$ of a length $L$ and denoting by $n_k(L)$ the number of cars in the $k$th subinterval, the average value $\overline{n}(L)$ taken over all possible subintervals is

$$\overline{n}(L) = \frac{1}{\lfloor N/L \rfloor} \sum_{k=1}^{\lfloor N/L \rfloor} n_k(L) = L,$$

where the integer part $\lfloor N/L \rfloor$ stands for the number of all subintervals included in the interval $[0, N]$. Number variance $\Delta_n(L)$ is then defined as

$$\Delta_n(L) = \frac{1}{\lfloor N/L \rfloor} \sum_{k=1}^{\lfloor N/L \rfloor} (n_k(L) - L)^2$$

and represents (in a traffic instance) the statistical variance in the number of vehicles operating at the same time inside a fixed part of the road of a length $L$.

As is well known from random matrix theory the number variance can be explicitly derived from the relevant spacing density $p_\beta(r)$. The significant advantage is remarkable sensitivity of the number variance $\Delta_n(L)$ to any change in the probability density $p_\beta(r)$—i.e. to any change in the potential $U(r_1, r_2, \ldots, r_n)$ also. Whereas the number variance of independent events (or non-interacting elements) is the identity $\Delta_n(L) = L$, for a thermodynamical traffic gas with non-zero inverse temperature $\beta$ there has been numerically calculated (in [18]) a different behaviour, concretely: a linear dependence

$$\Delta_n(L) \approx \chi L + \gamma \tag{5}$$

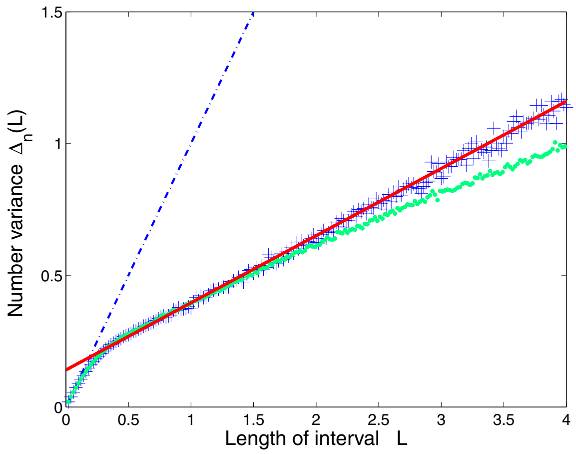


**Figure 4.** Results of the number-variance test. Plus signs stand for the value $\Delta_n(L)$ calculated for the collected traffic data, whereas points represent the same quantity for particles of Metropolis model (where $\beta_{\text{model}} = 1.45$). Dash-dotted line visualizes function $\Delta_n(L) = L$ representing the number variance of independent events. The solid curve displays function (5) calculated for the optimal value $\beta_{\text{fit}} \approx 1.2488$ obtained by the $\chi^2$-fit to the traffic data.

with a slope

$$\chi \approx \frac{1}{2.4360\,\beta^{0.8207} + 1} \leqslant 1$$

and a shift

$$\gamma \approx \frac{\beta}{5.1926\,\beta + 2.3929} \geqslant 0.$$

As understandable now, the comparison between the number variance of the collected data and function (5) can then be used (together with the comparison of the relevant gap distributions) as a robust fitting procedure which is capable of revealing more detailed nuances among the distributions compared. If applied to our topic, such a procedure generates the optimal value (4) for which the exactly determined number variance (5) corresponds to the measured data (see figure 4). Note that both of these curves $\Delta_n(L)$ are rapidly deflected from the line visualizing the number variance of non-interacting particles. It implies the presence of a strong repulsion among the vehicles. However, a small deviation is detected for larger $L$ between the traffic data (plus signs in figure 4) and Metropolis data (points in the same figure). Such a discrepancy can be explained by the simple fact that the respective temperatures (i.e. $\beta_{\text{model}}$ and $\beta_{\text{fit}}$) differ each from other.

## 6. Summary and discussion

The traffic ensemble of vehicles waiting at a red-light-signal on signal-controlled crossroads was investigated. We have introduced the thermal space-continuous time-discrete traffic model of repulsing point-like elements based on the Metropolis algorithm. By the suitable choice of the inverse temperature parameter there were obtained the same statistical distributions as those produced by the real traffic process. Above that, we show that the investigated state of the realistic traffic sample can be predicted with the help of the thermal-equilibrium state for local thermodynamical gas whose point-like particles are repulsed by the short-range power-law potential (1). As demonstrated above, for the fitted value $\beta \approx 1.25$ of reciprocal temperature the corresponding spacing distributions are practically the same. The correspondence between the traffic samples and presented theory is, moreover, supported by the robust test of number variance which reveals

(1) the thermal feature of the topic—on microscopic scale;
(2) the presence of strong interactions among the cars;
(3) a deep connection between the stationary state of waiting cars and the preceding move of the sample towards the intersection threshold.

To conclude, we assert that the configuration of vehicles waiting at a red-light-signal on signal-controlled crossroads is a product of local thermodynamics-like processes acting among the cars. All the accessible statistical analyses strongly support this fact. Therefore, the observed phenomenon can be understood as traffic in an especially super-congested state.

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## References

[1] Krbálek M 2007 *J. Phys. A: Math. Theory* **40** 5813
[2] Helbing D 2001 *Rev. Mod. Phys.* **73** 1067
[3] Schreckenberg M and Sharma S D 2001 *Pedestrian and Evacuation Dynamics* (Berlin: Springer)
[4] Helbing D, Farkas I and Vicsek T 2000 *Nature* **407** 487
[5] Abul-Magd A Y 2006 *Physica* A **368** 536
[6] Rawal S and Rodgers G J 2005 *Physica* A **346** 621
[7] Paromtchik I E and Laugier C 1996 *Proc. IEEE Intelligent Vehicle Symp., Tokyo, Japan* p 13
[8] http://math.arizona.edu/~rcan/Parking_Gap_Size.ppt
[9] Krbálek M and Šeba P 2000 *J. Phys. A: Math. Gen.* **33** L229
[10] Baik J, Borodin A, Deift P and Suidan T 2006 *J. Phys. A: Math. Gen.* **39** 8965
[11] Fouladvand M E, Sadjadi Z and Shaebani M R 2004 *J. Phys. A: Math. Gen.* **37** 561
[12] Mehta M L 1991 *Random Matrices (Revised and Enlarged)* (New York: Academic)
[13] Bogomolny E, Bohigas O, Leboeuf P and Monastra A G 2006 *J. Phys. A: Math. Gen.* **39** 10743
[14] Krbálek M and Helbing D 2004 *Physica* A **333** 370
[15] Metropolis N, Rosenbluth A W, Rosenbluth M N, Teller A H and Teller E 1953 *J. Chem. Phys.* **21** 1087
[16] Scharf R and Izrailev F M 1990 *J. Phys. A: Math. Gen* **23** 963
[17] Landau D P and Binder K 2005 *A Guide to Monte Carlo Simulations in Statistical Physics* 2nd edn (Cambridge: Cambridge University Press)
[18] Krbálek M and Šeba P 2008 *Preprint* physics/0611049 (submitted)